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Generalization of Lagrangian Dynamics

B. V. Pomorišac

Laboratory for Theoretical Physics-020, Institute "Boris Kidrič", P. O. Box 522, 11001 Beograd, Yugoslavia

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We speculate on a generalized dynamics described by an integral over action functionals that is a generalization of the standard functional integral. In a simple Gaussian case we obtain a certain differential equation for the measure of Feynman integral. We prove that the equation is satisfied for the spin zero field in one space-time dimension.

1. INTRODUCTION

Usually the Lagrangian is considered to be a basic object from which all of the dynamics follows; the form of the Lagrangian is normally assumed invariable. Here we adopt a different point of view—that the Lagrangian may be a dynamical entity subject to a certain equation. If such an equation is found it may provide means for systematization of different Lagrangians.

The stability of Lagrangians with respect to symmetry-breaking perturbations has been discussed recently (Foerster et al., 1980; Iliopulos et al., 1980). These papers did not consider arbitrary local variation of Lagrangian; consequently, the notion of stability used was somewhat special. If one wants to discuss stability with respect to arbitrary local variation, one may proceed in analogy to the way one defines stability for the solutions of Lagrangian equations. There one starts from a variational principle. The first variation gives the equation, while positivity of the second variation ensures local stability of the solution. Analogously, if one wants to discuss local stability of Lagrangian, it would be convenient to start from some kind of variational principle. Roughly, stability means here that predictions of the theory do not change very much with a small variation of Lagrangian.

However, we know that the variational principle of classical physics could be derived as a quasiclassical approximation to the full quantum

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problem (Feynman, 1948; Iliopulos et al., 1975). The standard way to derive this utilizes a steepest-descent approximation to some functional integrals. Likewise, for Lagrangians one may try to define an integral over functionals, then proceed with the quasiclassical approximation to find the equation that the Lagrangian should satisfy, and finally, check the stability of the solution. Here we follow this procedure for the simplest system—a one-dimensional scalar field.

In Section 2 an integral over functionals is defined, while in Section 3 a quasiclassical approximation is discussed and a particular solution to the resulting equation presented. After a brief discussion of the results, an appendix is devoted to the technical question of stability of the integral with respect to change of boundaries.

2. INTEGRAL OVER FUNCTIONALS

To define an integral over functionals let us recall the relation between the ordinary Riemann integral and the functional integral (hereafter we denote the former by I_1 and the latter by I_2, the convenience of this notation will be clear shortly). Roughly speaking, in I_1 some function $f(x)$ gives a weight with which point x enters the calculation. In the case of the functional integral I_2 used in quantum field theory, one integrates over $f(x)$, the weights of the integral I_1. Here point f enters the calculation with the weight $e^{iL(f)/\hbar}$, or with $e^{-L_E(f)/\hbar}$ in the Euclidean version of the theory (L_E being $-i$ times Lagrangian with Euclidean rotation). It is natural to ask the question how to define I_3, and what kind of dynamics (if any) can be described by it?

In defining I_3 we rely on analogy with I_2; I_3 is to I_2 as I_2 is to I_1. Thus in the integral I_3 we integrate over the weights of I_2, i.e., over functionals.

Since the consideration of the general I_3 may be very complicated, to get an insight into these ideas we discuss a particular, simple example. Consider functionals of bounded functions of one variable, $-k \leqslant \Phi(x) \leq k$, $-T \leqslant x \leqslant T$. Divide the interval $[-k, k]$ in N subintervals of length ε. We shall now enumerate integration variables. Divide the interval of possible values of function Φ at x_i in N units of length ε' (a particular division is inessential if the continuum limit exists). There are N possible values of Φ at x_1, N independent values at x_2, etc., up to x_N. We enumerate values of the function Φ in the discrete approximation with the set of integers $(r_1, r_2, \ldots, r_N) \equiv \mathbf{r}$. This $\mathbf{r}$ corresponds to function Φ with values $\Phi_i = -k + r_i\varepsilon$ at x_i. In this approximation there are N^N different functions. At each of the functions one may have a different value of the functional $G_{\mathbf{r}}$. To integrate over all functionals one should integrate separately over values of G taken at different $\mathbf{r}$. Thus one has, before taking the continuum limit, N^N integration variables. This is analogous to the standard definition of the functional integral where one has N integration variables. We define an I_3 by

$$I_3 = \int \delta G e^{-\beta M[G]}$$

$$\equiv \lim_{N\to\infty} \left(\prod_{\substack{1 \leqslant r_i \leqslant N \\ (1 \leqslant i \leqslant N)}} \int_{-\infty}^{\infty} \frac{dG_{r_1 \cdots r_N}}{B} \right) e^{-\beta \sum_{r_1 \cdots r_N} [(D_\Phi G_{\mathbf{r}}^2) + g G_{\mathbf{r}}^2] \frac{\varepsilon'^N}{A^N}}$$

$$\mathbf{r} \equiv (r_1, r_2, \ldots, r_N), \qquad N\varepsilon = 2T, N\varepsilon' = 2k \tag{1}$$

where

$$M[G] \equiv \int \delta\Phi \left\{ [D_\Phi G(\Phi)]^2 + g G^2(\Phi) \right\}$$

is the "hyperaction." Here the coefficient A stands for normalization factor of I_2 integral, while B denotes similar factor for I_3. To define operator D_Φ we are again led by the analogy with I_2—we take D_Φ that is inverse to (indefinite) functional integral:

$$\int \delta\Phi (D_\Phi G)^2 \equiv \lim_{N\to\infty} \int \cdots \int \frac{d\Phi_1}{A} \cdots \frac{d\Phi_N}{A} \times \left[\left(\prod_{i=1}^{N} C \frac{\partial}{\partial \Phi_i} \right) G(\Phi_1, \Phi_2, \ldots, \Phi_N) \right]^2$$

$$D_f \int_{\Phi < f} \delta\Phi F(\Phi) = F(f), \qquad \Phi_i \equiv \Phi(x_i) \tag{2}$$

Here C is a real constant, determined by the condition that limit $N \to \infty$ exists. We limit the space of functionals G to real functionals; with identification of G with the weight of I_2 it means that we use Euclidean weights.

An integral I_2 may be restricted at the boundary, here at values of G taken at $\Phi = -k, \Phi = k$. This restriction determines the boundary condition on functional G; it will not be explicitly written here.

The form of D_Φ is essential to ensure that I_3 is changed infinitesimally with the infinitesimal change of the boundary function. This is a necessary condition for the existence of $N \to \infty$ limit; it is discussed, at some length, in the appendix. This form of D_Φ also enables us to treat the quasiclassical approximation analogously as it is done for ordinary functional integral (Feynman, 1948; Iliopulos et al., 1975).

3. QUASICLASSICAL APPROXIMATION

To derive an equation for G consider a quasiclassical approximation to I_3 integral. In general, this approximation is justified in the case of low effective temperature $1/\beta$. For high β, the dominant contribution to I_3 is given by functionals that minimize $M[G]$. In the particular case of Gaussian integral like the one defined in Eq. (1) dependence on β is spurious; β can be scaled away by rescaling of G and the constant B and there are no corrections to quasiclassical results (see Iliopulos et al., 1975).

Taking $G \to G + \eta\delta(\Phi - \phi)$, $-k < \phi(x) < k$, in M defined by (1) and letting $\delta M = 0$ to first order in η gives

$$D^2_{\Phi,-\phi} G(\phi) + gG(\phi) = 0 \tag{3}$$

Here $D^2_{\Phi,-\phi}$ is defined by

$$D^2_{\Phi,-\phi} G(\phi) \equiv \lim_{N\to\infty} \left(\prod_{i=1}^{N} -C^2 \frac{\partial^2}{\partial\phi_i^2} \right) G(\phi_1, \phi_2, \ldots, \phi_N) \tag{4}$$

In the case that I_3 is restricted at the boundary Eq. (3) is supplemented by appropriate boundary conditions.

It is clear that acting of $D^2_{\Phi,-\phi}$ (and D_Φ) on any finite power of ϕ, e.g., of the form $\int K(x_1, x_2, \ldots, x_n)\phi(x_1)\phi(x_2)\cdots\phi(x_n)dx_1\cdots dx_n$ gives zero. One may look for the solution to Eq. (3) of the form $G(\phi) = e^{\alpha F(\phi)}$, where F may be a finite power of ϕ and its derivatives. It seems plausible that negative $\alpha F(\phi)$ would minimize M, rather than positive. Therefore, having in mind that we integrate over the weight of I_2, we identify $\alpha F(\phi)$ with $-L_E/\hbar$. Thus we assume that the whole Feynman measure satisfies Eq. (3). We now show that this holds for the spin zero field in one space-time dimension, i.e., for the Feynman measure of the one-dimensional quantum mechanical particle.

Let us start with the free, massless field:

$$G(\phi) = \exp(-L_E(\phi)/\hbar) = \exp\left[-\frac{1}{2h}\int_{-T}^{T}\left(\frac{d\phi}{dx}\right)^2 dx\right]$$

$$\simeq \exp\left[-\frac{1}{2\hbar}\sum_{i=1}^{N}\left(\frac{\phi_{i+1}-\phi_i}{\varepsilon}\right)^2 \varepsilon\right] \tag{5}$$

To calculate $D^2_{\phi,-\phi}$ one may vary independently each ϕ_i by ε_i and then find 2^N times the coefficient of the $\varepsilon_1^2\varepsilon_2^2\cdots\varepsilon_N^2$ term in the expansion of

$$
\begin{aligned}
&G(\phi_1+\varepsilon_1,\phi_2+\varepsilon_2,\ldots,\phi_N+\varepsilon_N):\ G(\phi_1+\varepsilon_1,\ldots,\phi_N+\varepsilon_N)\\
&\quad=\exp\left(-\frac{1}{2\hbar\varepsilon}\sum_{j=1}^{N-1}\Delta_j^2\right)\\
&\quad\times\exp\left\{\frac{1}{\varepsilon\hbar}\left[(\Delta_1\varepsilon_1-\Delta_{N-1}\varepsilon_N)-\frac{1}{2}\varepsilon_1^2-\frac{1}{2}\varepsilon_N^2+\varepsilon_1\varepsilon_2\right]\right\}\\
&\quad\times\prod_{j=2}^{N-1}\left\{\exp\left[\frac{1}{\varepsilon\hbar}(\Delta_j-\Delta_{j-1})\varepsilon_j-\frac{1}{\varepsilon\hbar}\varepsilon_j^2+\frac{1}{\varepsilon\hbar}\varepsilon_{j+1}\varepsilon_j\right]\right\}\\
&\qquad\qquad\qquad\Delta_j\equiv\phi_{j+1}-\phi_j
\end{aligned}
\tag{6}
$$

We assume that the function ϕ belongs to a space of sufficiently smooth functions, specifically, that $[(\Delta_j-\Delta_{j-1})/\varepsilon]^2\to 0$; for this it is sufficient that the first derivative exists. [By detailed analysis one can show that this condition is not essential for validity of Eq. (3) in this case; the recursion relation (7) holds asymptotically for $k\to\infty$ due to the factor $\exp(-1/2\varepsilon\hbar\Sigma\Delta_j^2)$, but the calculated constant g may change.]

Denoting the coefficient of $\varepsilon_1^2\varepsilon_2^2\cdots\varepsilon_k^2$ term by c_k and expanding the exponent in (6) up to ε_j^2 and $(\varepsilon_{j+1}\varepsilon_j)^2$ one obtains, to the leading order in ε the recursion relation

$$
\begin{aligned}
&d_{k+1}=\tfrac{1}{2}d_{k-1}+d_k,\qquad d_1=\tfrac{1}{2},\qquad d_2=1\\
&d_k\equiv(-\varepsilon\hbar)^k c_k
\end{aligned}
\tag{7}
$$

On defining $R_k\equiv d_k/d_{k-1}$ one easily shows that R_k make rapidly converging Cauchy sequence and $\lim_{k\to\infty}R_k=(1+\sqrt{3})/2$. This ensures the power behavior of c_k,

$$
c_k\underset{k\to\infty}{\sim}\left(\frac{-1}{\hbar\varepsilon}\right)^k\left(\frac{1+\sqrt{3}}{2}\right)^k\cdot\frac{1}{2}
$$

where the last factor $1/2=\lim_{k\to\infty}d_k/[(1+\sqrt{3})/2]^k$ was calculated numerically. Power behavior of c_k enables us to determine C^2 given by Eq. (4), such that $D^2_{\phi,-\phi}G(\phi)$ exists. [This is not trivial; for example, $D_\phi G(\phi)$ does not exists.] Taking $C^2=\varepsilon\hbar/1+\sqrt{3}$ one obtains

$$
D^2_{\phi,-\phi}\exp\left[-\frac{1}{2\hbar}\int_{-T}^{T}\left(\frac{d\phi}{dx}\right)^2dx\right]=\frac{3-\sqrt{3}}{4}\exp\left[-\frac{1}{2\hbar}\int_{-T}^{T}\left(\frac{d\phi}{dx}\right)^2dx\right]\tag{8}
$$

With some labor one can see that adding a local self-interaction term to

$L_E(\phi)$ does not affect Eq. (8). In effect, adding the term

$$\int_{-T}^{T} V(\phi)\, dx \simeq \sum_{j=1}^{N-1} V\left(\frac{\phi_j + \phi_{j+1}}{2}\right)\varepsilon$$

to L_E brings in terms of the same order as terms containing Δ_j in Eq. (6). Assuming differentiability of $V(\phi)$, one can write the recursion relations for this case and show that to leading order in ε one obtains Eq. (7). Thus Eq. (3), with a particular value of g given by (8), is solved by a large class of functionals G_0; $G_0 = \exp[-(1/\hbar)L_E]$, $L_E = \int_{-T}^{T}[\frac{1}{2}(d\phi/dx)^2 + V(\phi)]dx$. This large class of solutions to Eq. (3) can be reduced by imposing boundary conditions at $\phi = \pm k$. For example, if one demands $G(\phi = k) = G(\phi = -k) = 1$ and G that does not explicitly depend on k, one obtains $V(\phi) = 0$—a free massless field.

4. DISCUSSION

As one sees from Eq. (8) the solution to Eq. (3) is valid for a particular value of g, $g = (\sqrt{3} - 3)/4 \simeq -0.317$. Constant g is negative, and that creates two (related) problems. First, it is not clear that the term $(D_\phi G)^2$ suffice for convergence of I_3 in this case, and second, G_0 does not minimize $M[G]$ since adding a constant to G_0 can decrease M; that is, the solution is not stable. Imposing boundary conditions on G may alleviate both of these problems. First, since G fixed at boundary cannot become overly large without making the term $(D_\phi G)^2$ large; second, since adding a constant to G would spoil boundary conditions. Adding a term of the form $\int \delta\phi G^4$ would also alleviate the problems; however, this would change Eq. (3) and finding a nontrivial solution would become difficult.

We have used an I_3 integral to derive a possible form of the equation for the measure of Feynman integral. However, if one wants to make full analogy with the dynamics described by I_2, one would have to integrate all standard expressions with respect to $G = e^{-L_E/\hbar}$, and to consider invariable L_E as a quasiclassical approximation to full expression. Since we observe rather constant Lagrangians, it is clear that the parameter β should be large, i.e., that we are dealing with a low effective "temperature" system. It is tempting to speculate further that β may represent the inverse temperature of the universe, taken in suitable units. Thus one would have an ensemble of interactions, of which only the ones that minimize M are visible at present. One then may consider models similar to the work of Foester et al. and Iliopulos et al. (1980) where minimum of M has the highest symmetry.

Another model that one may think of is to use $M[G]$ with more than one minimum. Expanding about each minimum one would get free Lagrangians, while tunneling may describe interactions. In this manner one may describe different particles by one field.

APPENDIX

If the integral I_3 exists, it should change infinitesimally with the infinitesimal change of the function where the boundary condition on G is imposed. Impose a boundary condition of the form $G_{f_2} = G_2$, where f_2 is given by $f_2(x_i) = -T + N_i\varepsilon'$. Here we are using an arbitrary, nonconstant function f_2—analogous derivation can be given for f_1. In the discrete notation the boundary condition is written $G_{N_1N_2\cdots N_N} = G_2$ and one does not integrate over this variable. We write operator D_ϕ in the discrete notation:

$$\delta_j G_{r_1r_2\cdots r_j\cdots r_N} \equiv G_{r_1\cdots r_j\cdots r_N} - G_{r_1\cdots r_j-1\cdots r_N}$$

$$D_\phi G_\phi = \prod_{j\leq N} C\left(\frac{\delta_j}{\varepsilon'}\right) G_{\mathbf{r}}$$

With this we define I_3 by

$$I_3(f_2, G_2) \equiv \lim_{\substack{N\to\infty\\ N_i\to\infty}} \left(\prod_{r_k\leq N_k}' \int \frac{dG_{r_1r_2\cdots r_N}}{B}\right)$$

$$\times \exp\left(-\beta \sum_{r_i\leq N_i}\left\{\left[\left(\prod_{j\leq N} C\frac{\delta_j}{\varepsilon'}\right)G_{\mathbf{r}}\right]^2 + gG_{\mathbf{r}}^2\right\}\left(\frac{\varepsilon'}{A}\right)^N\right)\Bigg|_{G_{N_1N_2\cdots N_N} = G_2} \tag{A1}$$

Here $\prod'$ means that we exclude the term with all the indices at the highest value, i.e., $\sum_{k=1}^N r_k < \sum_{k=1}^N N_k$.

Now we derive the relation between variables $G_{\mathbf{r}}$ and $(\prod_{j\leq N}\delta_j)G_{\mathbf{r}}$. Define $G_{\mathbf{r}}$ with any index $r_i = 0$ to be certain constant G_1, that is, $\delta_i G_{r_1\cdots r_i=1\cdots r_N} = G_{r_1\cdots r_i=1\cdots r_N} - G_1$. Then one can show

$$G_{r_1r_2\cdots r_N} = G_1 + \sum_{r_i'\leq r_i}\left(\prod_{j\leq N}\delta_j\right)G_{\mathbf{r}'} \tag{A2}$$

This can be shown by induction in the number of dimensions of the vector

r. For one dimensional r one has

$$\sum_{r_i' \leqslant r_i} (\Pi \delta_j) G_{\mathbf{r}'} = \sum_{r_1' \leqslant r_1} \delta_1 G_{r_1'} = (G_{r_1} - G_{r_1 - 1}) + (G_{r_1 - 1} - G_{r_1 - 2}) + \cdots (G_{r_1 = 1} - G_1) = G_{r_1} - G_1$$

Using the formula for k-dimensional r one easily goes to $(k+1)$-dimensional r. From (A2) follows

$$G_{r_1 \cdots r_s = N_s + 1 \cdots r_N} = G_{r_1 \cdots r_s = N_s \cdots r_N} + \sum_{\substack{r_s' = N_s + 1 \\ r_{i \neq s}' \leqslant r_i}} \left(\prod_{j \leqslant N} \delta_j \right) G_{\mathbf{r}'} \tag{A3}$$

One can also show, using the induction in the number of dimensions of r, that the Jacobian of the transformation from $G_{\mathbf{r}}$ to $(\prod_{j \leqslant N} \delta_j) G_{\mathbf{r}}$ is 1.

If one changes infinitesimally the function f_2 by

$$f_2(x_i) \to f_2'(x_i) = -T + N_i \varepsilon' + \delta_{is} \varepsilon'$$

or in the discrete notation

$$\mathbf{r}_{f_2} \to \mathbf{r}_{f_2}' = (N_1, N_2 \cdots N_s + 1, \ldots, N_N)$$

the induced change in I_3 must be of the order ε':

$$I_3(f_2', G_2)$$

$$= \lim_{\substack{N \to \infty \\ N_i \to \infty}} \left(\prod_{r_k \leqslant N_k}' \int \frac{dG_{\mathbf{r}}}{B} \right) \int \frac{dG_{N_1 \cdots N_s \cdots N_N}}{B} \left(\prod_{\substack{r_{k \neq s} \leqslant N_k \\ r_s = N_s + 1}}' \int \frac{dG_{r_1 \cdots N_s + 1 \cdots r_N}}{B} \right)$$

$$\times \exp\left(-\beta \sum_{\substack{r_s = N_s + 1 \\ r_{i \neq s} \leqslant N_i}} \left\{ \left[\left(\prod_{j \leqslant N} C \frac{\delta_j}{\varepsilon'} \right) G_{\mathbf{r}} \right]^2 + g G_{\mathbf{r}}^2 \right\} \left(\frac{\varepsilon'}{A} \right)^N \right)$$

$$\times \exp\left(-\beta \sum_{r_k \leqslant N_k} \left\{ \left[\left(\prod_{j \leqslant N} C \frac{\delta_j}{\varepsilon'} \right) G_{\mathbf{r}} \right]^2 + g G_{\mathbf{r}}^2 \right\} \left(\frac{\varepsilon'}{A} \right)^N \right) \Bigg|_{G_{N_1 \cdots N_s + 1 \cdots N_N} = G_2} \tag{A4}$$

As $\varepsilon' \to 0$ left side of this equation tends to $I_3(f_2, G_2)$. To ensure that the right side tends to expression given by (A1) one needs a set of δ functions:

$\delta(G_{r_1 r_2 \cdots N_s+1 \cdots r_N} - G_{r_1 r_2 \cdots N_s \cdots r_N})$, $r_{i\neq s} \leqslant N_i$. This is provided by the term

$$\exp\left(-\beta \sum_{\substack{r_s = N_s+1 \\ r_{i\neq s} \leqslant N_i}} \left\{ \left[\left(\prod_{j \leqslant N} C \frac{\delta_j}{\varepsilon'} \right) G_{\mathrm{r}} \right]^2 \right\} \left(\frac{\varepsilon'}{A} \right)^N \right)$$

Here to zeroth order in ε' the term gG_{r}^2 does not contribute. By changing the variables from

$$G_{r_1 \cdots N_s+1 \cdots r'_N} G_{N_1 \cdots N_s \cdots N_N}$$

to

$$Y_{r_1 \cdots N_s+1 \cdots r_N} = (\Pi \delta_j) G_{r_1 \cdots N_s+1 \cdots r_N}$$

one shows that normalization of the δ functions requires

$$B = \frac{\left[\pi(A\varepsilon')^N\right]^{1/2}}{(\beta C^N)^{1/2}}$$

The condition $G_{N_1 \cdots N_s+1 \cdots N_N = G_2}$ becomes

$$G_{N_1 \cdots N_s \cdots N_N} = G_2 + \sum_{\substack{r'_{i\neq s} \leqslant N_i \\ r'_s = N_s+1}} Y_{r'_1 \cdots N_s+1 \cdots r'_N}$$

In the integral y's are of the order $(\varepsilon' A)^{N/2}$. In the correction the liner term in y vanishes due to parity of the Gaussian and the correction is, indeed, of the order ε'. This is analogous to the derivation of the Schrödinger equation from the functional integral (Feynman, 1948). However, here the first-order correction does not give closed equation for $I_3(f_2, G_2)$—it only expresses it through another I_3.

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REFERENCES

Feynman, R. P. (1948). *Reviews of Modern Physics*, **20**, 367.
Foerster, D., Nielsen, H. B., and Niomiya, M. (1980). *Physics Letters*, **94B**, 135.
Iliopulos, J., Itzykson, C., and Martin, A. (1975). *Reviews of Modern Physics*, **47**, 165.
Iliopulos, J., Nanopulos, D. V., and Tomaras, T. N. (1980). *Physics Letters*, **94B**, 141.